HELICAL HYDROMAGNETIC DYNAMO

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1. Examples of the self-excitation of a magnetic field [1-6] are of interest to hydromagnetic dynamo theory. In the helical model of a dynamo the field is excited by discontinuous axisymmetric motion [5, 6]. The case in which the axisymmetric motion may be continuous is discussed below.

The transition to continuous motion is accompanied by the emergence of a large number of solutions of the dynamo problem which increase in time.

The induction equation (for a medium with a magnetic viscosity equal to unity) is

$$\partial \mathbf{H}/\partial t = \mathrm{rot}\,[\mathbf{U}\times\mathbf{H}] + \Delta\mathbf{H}, \tag{1.1}$$

where the velocity U is given.

Let the velocity components in the cylindrical coordinates r, φ, and z be $U_r = 0$, $U_\varphi = r\omega_*(r)$, and $U_z = v_*(r)$. Assuming the field to be proportional to $\exp(im\varphi + ikz + pt)$, we obtain from (1.1)

$$\begin{gathered}\left(rDrD - m^2 - 1 \mp 2m - r^2 q_*^2\right) H_\pm = \mp {}^1/_2 ir^3 (H_+ + H_-) D\omega_* \\ \left(D = d/dr,\quad q_*^2 = s^2 + i\mu_*,\quad s^2 = p + k^2,\quad \mu_* = m\omega_* + kv_*\right)\end{gathered} \tag{1.2}$$

for the quantities $H_\pm = H_r + iH_\varphi$.

The quantities

$$H_\pm,\ DH_\pm \pm {}^1/_2 i\omega_*(H_+ + H_-) \tag{1.3}$$

should be continuous [5] on the discontinuity surface of the velocity or its derivative.

The solutions of the problem (1.2) and (1.3) should be continuous and vanish as $r \to \infty$. A field is generated if an eigenvalue p with a positive increment $\gamma = \mathrm{Re}\,p$ exists. Generation is impossible if one of the parameters m, k, or v_* is equal to zero [2, 7].

Let us assume that in some range of values of r

$$\omega_* = \omega + \Omega/r^2,\ v_* = v + V/r^2, \tag{1.4}$$

where ω, Ω, v, and V are constants. We set

$$\begin{gathered}\mu = m\omega + kv,\ M = m\Omega + kV,\ q^2 = s^2 + i\mu, \\ N = (m^2 + im\Omega)^{1/2}(|\arg N| < {}^1/_4\pi).\end{gathered} \tag{1.5}$$

Then one can represent the solution of (1.2) in the form

$$H_\pm = A_\pm I_\pm + B_\pm K_\pm - \frac{N-m}{N+m}(A_\mp I_\mp + B_\mp K_\mp), \tag{1.6}$$

where $A_\pm$ and $B_\pm$ are constants and $I_\pm$ and $K_\pm$ are the modified Bessel functions of argument qr with indices

$$\nu_\pm = (1 + m^2 + iM \pm 2N)^{1/2}. \tag{1.7}$$

Let us consider the case in which only the constants ω and v in (1.4) and (1.5) are different from zero for r < 1 and only Ω and V are different from zero for r > 1. Taking $B_\pm = 0$ for r < 1 and $A_\pm = 0$ for r > 1 in (1.6), one can derive from (1.3) the dispersion relation

$$\begin{gathered}2i(\omega - \Omega)\left(j_- - \frac{m}{M}k_-\right) = k_+^2 - k_-^2 + j_+^2 - j_-^2\ 2k_+j_+ + \frac{m^2+N^2}{mN}k_-j_- \\ \left(j_\pm = \frac{qI'_+(q)}{I_+(q)} \pm \frac{qI'_-(q)}{I_-(q)},\quad k_\pm = \frac{sK'_+(s)}{K_+(s)} \pm \frac{sK'_-(s)}{K_-(s)},\quad |\arg s| < \pi/2\right).\end{gathered} \tag{1.8}$$

This relation has been investigated in [5] for $\Omega = V = 0$ and large ω, q, and s. The case in which Ω and V may also be large is discussed below.

2. We will simplify (1.8) by assuming q and s to be large. The values of the quantities

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$$j_+ = 2q,\ j_- = 2m/q \quad (|q| \gg m,\ |\arg q| < {}^1/_2\pi) \tag{2.1}$$

are obtained from the asymptotes [5, 8] of the functions $I_\pm$ with fixed indices $m \pm 1$.

It is necessary for the determination of $k_\pm$ to use the asymptote of $K_\nu(s)$ with arbitrary values of ν/s, since the indices (1.7) increase without limit as Ω and V increase.

One can derive the necessary asymptote from the equation [8]

$$K_\nu(s) = {}^1/_2 \int_{-\infty}^{\infty} \exp(\nu t - s \operatorname{ch} t)\, dt \quad (|\arg s| < {}^1/_2\pi) \tag{2.2}$$

by the method of steepest descent.

We denote $\arg \nu = \psi$. Since K_ν is an even function of ν, it is possible to assume $|\psi| \leqslant {}^1/_2\pi$. It is sufficient to consider the case $\psi \geq 0$, since a complex-conjugate value of the function (2.2) is obtained upon the replacement of ν and s by their complex-conjugate values.

For $0 \leqslant \psi \leqslant {}^1/_2\pi$ the contour of steepest descent can pass through the saddle points

$$t_\pm = i\pi/2 \pm (\alpha + i\beta) \quad (\alpha \geqslant 0,\ \operatorname{sh} t_\pm = \nu/s,\ {}^1/_2\pi \geqslant \beta \geqslant 0)$$

with contributions to the integral (2.2) which are equal in absolute value.

The values of the quantity

$$\xi = s^2/\nu^2 = \rho e^{i\theta} \quad (-\pi - 2\psi \leqslant \theta \leqslant \pi - 2\psi)$$

for which these two conditions are satisfied lie on the line of zeros [8] of the function (2.2).

The contributions of the points $t_\pm$ are given by the expressions

$$\varkappa_\pm = \sqrt{\frac{\pi}{2\nu\delta}} \exp i({}^1/_2\pi\nu + {}^1/_4\pi \pm \eta), \tag{2.3}$$
$$\eta = (\nu/i)(\ln(1+\delta) - {}^1/_2 \ln \xi - \delta - {}^1/_2\pi i) - {}^1/_4\pi,$$
$$\delta = \sqrt{1+\xi},$$

in which all the radicals and logarithms are positive for positive values of ν and ξ. It is assumed that

$$|\nu\delta| \gg 1,\ |\nu\delta^3| \gg 1. \tag{2.4}$$

The contributions of (2.3) are equal in absolute value if $\operatorname{Im} \eta = 0$. The solution $r(\theta)$ of this equation in the region $r \leq 1$ and $-2\psi - \pi \leq \theta \leq -\pi$ corresponds to the line of zeros Γ. The limiting expressions of the line are

$$\arg \delta = {}^1/_3(\pi/2 - \psi) \quad (|\delta| \ll 1), \tag{2.5}$$
$$\rho = 4 \exp[(\theta + \pi) \operatorname{tg} \psi - 2] \quad (\rho \ll 1).$$

For $\psi = {}^1/_2\pi$ the line Γ is the segment $-1 \leq \xi \leq 0$. The function (2.2) does not have zeros [8] on the ξ plane outside Γ. Near and on Γ the integral (2.2) is asymptotically equal to the sum of the contributions (2.3), and far from it – to the largest of the contributions.

We note that $\varkappa_\pm$ are the values of one and the same function on different sides of Γ; $\varkappa_-$ is obtained from $\varkappa_+$ by going clockwise around the point $\xi = -1$.

It follows from (2.3) that

$$\frac{sK_\nu'(s)}{K_\nu(s)} = -\nu\delta \begin{Bmatrix} i \operatorname{tg} \eta \\ 1 \end{Bmatrix}, \tag{2.6}$$

where the upper factor is taken near and on Γ and the lower one is taken far from it. In the case $\psi < 0$ it is necessary to multiply i by $\sigma = \operatorname{sgn} \psi$ in (2.3) and (2.6).

According to (2.6), outside the lines of zeros $\Gamma_\pm$ of the functions $K_\pm$

$$k_\pm = -\nu_+ (\sqrt{z+1} \pm \sqrt{z + \varepsilon^2}) \quad (z = s^2/\nu_+^2,\ \ \varepsilon = \nu_-/\nu_+). \tag{2.7}$$

Near Γ_- the second radical in (2.7) is replaced by $i\sigma\varepsilon\delta \tan \eta$, where now

$$\delta = \sqrt{z/\varepsilon^2 + 1},\ \sigma = \operatorname{sgn} \operatorname{Im}(\nu_+\varepsilon),$$
$$\eta = \frac{\nu_+\varepsilon}{i\sigma}\left[\ln(1+\delta) - \frac{1}{2}\ln\frac{z}{\varepsilon^2} - \delta - \frac{1}{2} i\pi\sigma\right] - \frac{1}{4}\pi. \tag{2.8}$$

In an analogous replacement of the first radical $\varepsilon = 1$. Substitution of (2.1) and (2.7) into (1.8) gives

$$\frac{\sqrt{z_*}\sqrt{z+1}\sqrt{z+\varepsilon^2}+z_*\left(\sqrt{z_*}+\sqrt{z+1}+\sqrt{z+\varepsilon^2}\right)-\left(\sqrt{z+1}-\sqrt{z+\varepsilon^2}\right)/(4E)}{1+E\sqrt{z_*}\left(\sqrt{z+1}-\sqrt{z+\varepsilon^2}\right)}$$
$$-\frac{m^2}{\nu_+^2\sqrt{z_*}}=\frac{im(\omega-\Omega)}{\nu_+^3}\equiv a \qquad \left(z_*=\frac{q^2}{\nu_+^2},\quad E=\frac{2}{1-\varepsilon^2}\right). \tag{2.9}$$

It is shown below that no more than one eigenvalue satisfies Eq. (2.9). One can call this value isolated. It exists only for large a and is characteristic of discontinuous motion. As a decreases, it vanishes, withdrawing into a branch cut of the z plane or becoming nonisolated, approaching one of the lines $\Gamma_\pm$.

Nonisolated eigenvalues exist for small a. They are located near $\Gamma_\pm$. The distance between adjacent values of z decreases without bound as the velocities increase. The increments of these values can be positive if $|\arg \nu_\pm| > {}^1/_4\pi$; according to (1.7), this condition can be satisfied only for ν_-.

3. First we will consider the case μ = M = 0. Assuming Ω to be large, we adopt

$$\varepsilon = -i \operatorname{sgn}\Omega,\ |\nu_+| \gg m,\ \arg\nu_+ \approx {}^1/_8\pi \operatorname{sgn}\Omega \tag{3.1}$$

in (2.7)-(2.9).

Neglecting the second fraction in (2.9), we obtain

$$2a = (2\sqrt{z}-\sqrt{z+1}-\sqrt{z-1})^{-1}. \tag{3.2}$$

This equation is solvable in radicals, since it is reduced to a cubic equation by the substitution

$$\sqrt{z\pm 1} = (1/y \pm y)/\sqrt{2}\ .$$

It is simpler to obtain the particular values z(a) directly from (3.2).

With A = $|a| \gg 1$,

$$z=\left(\frac{1}{2}a\right)^{2/3}=\left(\frac{1}{2}m\,|\,\omega-\Omega\,|\right)^{2/3}\nu_+^{-2}\exp\left[\frac{1}{3}i\pi\operatorname{sgn}(\omega-\Omega)\right] \tag{3.3}$$

follows from (3.2). One can verify with the help of (2.9) that (3.3) is valid for small Ω (it has been found earlier [5] for Ω = 0). The values of μ and M may also turn out to differ from zero.

The determination of z for large Ω is facilitated by the fact that $\vartheta = \arg a$ varies weakly. For example,

$$\vartheta = {}^1/_2\pi - {}^3/_8\pi \operatorname{sgn}\Omega\ (\Omega < \omega,\ \omega > 0). \tag{3.4}$$

Let us find the value of a for which z > 0. We have from (3.2)

$$2a = (2\sqrt{z}-\sqrt{z+1}-i\sqrt{1-z})^{-1}\ (0 \leqslant z \leqslant 1).$$

From this it follows that

$$A^{-1} = 2\sqrt{2}(\sqrt{z+1}-\sqrt{z}),\ \operatorname{tg}\vartheta = \sqrt{1-z}/(2\sqrt{z}-\sqrt{z+1}).$$

Using (3.4) when $\Omega > 0$, we obtain z ≈ 0.9 and A = A_0 ≈ 0.8. For $\Omega < 0$ and z < 0 the result is

$$A^{-1} = 2\sqrt{2}(\sqrt{1-z}-\sqrt{-z}),\ \operatorname{tg}\vartheta = (2\sqrt{-z}-\sqrt{1-z})/\sqrt{1+z}\ (-1 \leqslant z \leqslant 0).$$

Thus it follows from (3.4) that z ≈ −0.1 and A ≈ 0.5.

The particular values of z found are used for the qualitative plotting of graphs of z(Ω). The lines of zeros $\Gamma_\pm$ and the branch cut of the z plane are shown† in Fig. 1 for large $\Omega < 0$; plots of z(Ω) as $|\Omega|$ increases from zero are shown by a solid curve for the case ω = const≫1 and by a dotted line for the case $\omega - \Omega$ = const≫1. Similar curves are shown in Fig. 2 for $\Omega > 0$.

According to (2.5) and (3.1), |z| ≈ 0.25 and |z| ≈ 0.001 at the intersection points of the lines of zeros with the branch cut.

It is evident in Fig. 1 that in the second case an isolated eigenvalue ceases to exist, withdrawing into the branch cut; then |z| ≈ 0.1 and A ≈ 0.4.

It is evident in Fig. 2 that in both cases when A is somewhat less than A_0 the eigenvalue approaches Γ_- and then remains there, differing not at all from the other eigenvalues near Γ_-. In the first case a new isolated eigenvalue arises which emerges from the branch cut; then $\Omega > \omega$, and the values |z| ≈ 0.1 and A ≈ 0.4 are equal to those found for $\Omega < 0$.

†The generally adopted coordinate system is used: The real and imaginary axes pass through zero horizontally to the right and vertically upward.

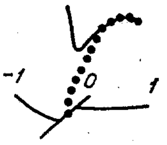

Fig. 1

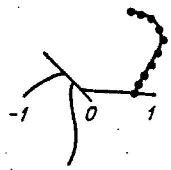

Fig. 2

4. As $|\Omega|$ increases, nonisolated eigenvalues emerge from the branch cut and draw apart from it, remaining near the lines of zeros.

We will restrict ourselves to consideration of the eigenvalues near Γ_-, since the increment is positive only for them. Making the replacement (2.7) with (3.1) taken into account, we obtain from (2.9)

$$\operatorname{tg}\eta = \frac{1}{\delta}\,\frac{z\sqrt{z+1} + z\sqrt{z} - {}^1/_4\sqrt{z+1} - a(\sqrt{z}\sqrt{z+1} + 1)}{\sqrt{z}\sqrt{z+1} + z + {}^1/_4 + a\sqrt{z}} \tag{4.1}$$

$$(\delta = \sqrt{1-z},\ \eta = \nu_+[\ln(1+\delta) - {}^1/_2 \ln z - \delta] - {}^1/_4\pi).$$

For small δ we find

$$\pi(n - {}^1/_2) = \eta = {}^1/_3\,\nu_+\delta^3 - {}^1/_4\pi \quad (|\delta| \ll 1,\ n = 1, 2, \ldots) \tag{4.2}$$

from (4.1). Thence

$$\delta = \left[3\pi i\left(n - {}^1/_4\right)(\operatorname{sgn}\operatorname{Im}\nu_-)/\nu_-\right]^{1/3}, \quad p = -k^2 - \nu_-^2 + \nu_-^2\,\delta^2 \tag{4.3}$$

follows from (3.1), (2.7), and (1.2).

It is evident from (4.2), (4.1), and (2.6) that the eigenvalues coincide with the zeros of K_- in the approximation (4.2). One can explain this coincidence as follows.

The left- and right-hand sides of (1.8) differ by some finite value if the derivative K_-'/K_- is equal to zero. It is evident from (2.6) that this derivative is small everywhere when δ is small except for a small neighborhood of the zeros of K_-. The derivative takes on any value in the neighborhood of each zero, including the value for which (1.8) is satisfied.

The inequalities (2.4) and (4.2) are satisfied for (4.3) if $|\nu_-|^{1/3} \gg n^{1/3} \gg 1$. When these conditions are violated, the deviation from the value determined from (4.3) can be of the order of the distance to the neighboring value.

In the case $n \sim |\nu_-|$ one can substitute the value of z determined from the first of Eqs. (4.1) into the right-hand side of (4.1).

In the case $n \sim 1$ it is necessary [8] to use Nicholson's approximation instead of (2.3). In this approximation the equation $K_- = 0$ is converted to the form

$$J_{1/3}(\tau) + J_{-1/3}(\tau) = 0 \quad (\tau = {}^1/_3\nu_+\delta^3).$$

Equation (4.2) is obtained from the above with $\tau \gg 1$. The roots $\tau = 2.38$ and 5.51 are also close to (4.2) for $n = 1.2$. Therefore (4.3) is valid for $n \sim 1$.

It was assumed above that $\mu = M = 0$. Equations (4.3) prove to be valid in the general case (their derivation from (2.9) and (2.8) is not altered).

It is evident from (4.3) and (1.7) that p does not depend upon μ and that

$$p(M) \approx -k^2 - \nu_-^2 = p(0) - iM \quad (|nM|^{3/2} \ll |N|).$$

For $M = M_* = 2\operatorname{Im} N$, the number p is real, and the increment

$$\gamma = b - k^2 - [3\pi(n - {}^1/_4)b]^{2/3} \quad (b = 2\operatorname{Re} N - m^2 - 1)$$

is a maximum and positive for $b > b_*(k, n)$. The critical value b_* increases as k and n increase. As $|M - M_*|$ increases, the increment drops off to $\gamma \approx \operatorname{Re}\nu_-^2\delta^2 < 0$ and generation is curtailed. The increment decreases as n increases.

The isolated eigenvalue satisfies the relationship

$$p(\mu, M) = p(0, M - \mu) - i\mu. \tag{4.4}$$

This follows from the preservation of Eqs. (2.7) and (2.9) upon the replacement in them of z and $\nu_{\pm}(M)$ by z_* and $\nu_{\pm}(M-\mu)$. According to (4.4), the general case reduces to the case $\mu = 0$, and the case $\mu = M$ reduces to the one investigated above. The value (4.4) may be real as is the case for (4.3).

Smoothing of the velocities was achieved above by decreasing the sizes of their discontinuities. Another smoothing method is the replacement of the discontinuity by a transition region $1 \le r \le r_0$ in which the velocities (1.4) change continuously from constant values for $r \le 1$ to zero for $r \ge r_0$. One can find the exact dispersion relationship in this case. One should expect that both smoothing methods lead to similar changes in the spectrum of eigenvalues.

The example considered shows that smoothing of the velocities can appreciably alter the spectrum of the dynamo problem.

The velocity distribution (1.4), for which Eqs. (1.2) are integrated exactly, was considered above. In the case of arbitrary velocities one should use the WKB approximation to search for the integrals of (1.2).

Many boundary-value problems are presently discussed in this approximation with a fourth-order equation (two second-order equations). One should note the review by Erokhin and Moiseev [9], in which recent achievements in this direction are summarized.

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